QUANTUM MECHANICS Lecture 6

Still about the Schrödinger equation

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D. J. Griffiths: paragraph 2.2

General solution of the Schrödinger equation

1. In the previous lecture we have said that the procedure to find the **general solution** of **the time dependent** Schrödinger equation, for an initial condition $\Psi(x,0)$, is as follows.
2. We solve the time-independent Schrödinger equation, which, in general, has infinite solutions $\psi_1(x)$, ..., $\psi_n(x)$, ... corresponding to different energies E_1, ..., E_n, ...
3. We write $\Psi(x,0)$ as a linear combination of the above stationary solutions, i.e.

$$\Psi(x,0) = \sum_n c_n \, \psi_n(x)$$

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General solution of Schrödinger equation

1. We define the function

$$\Psi(x,t) = \sum_n c_n \, e^{-iE_n t/\hbar} \, \psi_n(x)$$

2. Since it is a linear combination of solutions of the *time − dependent* Schrödinger equation, it is certainly one of its possible solutions.
3. At $t = 0$, the w.f. $\Psi(x,t)$ satisfies the initial condition that we have imposed, therefore **it is the solution that we were looking for**, ***because the solution with a given initial condition is unique.***

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About the time dependence

1. Let us come again, now, to the argument of the time dependence af the quantum observables . . . .

2. We have seen that each stationary state describes a physical state which appears "*frozen*" in time: no time dependence of any physical quantity !

3. But things **do** change in time . . . **how it happens ?**

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About the time dependence

1. Suppose that

$$\boldsymbol{\Psi}(x,0) = c_1\,\psi_1(x) + c_2\,\psi_2(x)$$

where, for sake of simplicity, c_1 and c_2 are real numbers such that $c_1^2 + c_2^2 = 1$ and ψ_1, ψ_2 are real *normalized* (orthogonal) stationary solutions, corresponding to the energies E_1 and E_2, with $E_1 \neq E_2$.

2. Then the time-dependent solution reads

$$\boldsymbol{\Psi}(x,t) = c_1\,\psi_1(x)\,e^{-iE_1 t/\hbar} + c_2\,\psi_2(x)\,e^{-iE_2 t/\hbar} \equiv$$
$$\equiv c_1\boldsymbol{\Psi}_1(x,t) + c_2\boldsymbol{\Psi}_2(x,t)$$

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About the time dependence

Let us see, for instance, how it behaves in time the expectation value of the position x.

If we define $\Delta E \equiv E_2 - E_1$, then we have

$$
\begin{aligned}
< x > &= \int dx\,(c_1\Psi_1 + c_2\Psi_2)^*\,x\,(c_1\Psi_1 + c_2\Psi_2) = \\
&= c_1^2 < x >_1 + c_2^2 < x >_2 + \\
&+ 2c_1\,c_2\,cos\left(\frac{\Delta E t}{\hbar}\right)\int dx\ \psi_1(x)^* \cdot \psi_2(x) \cdot x
\end{aligned}
$$

which shows that the expectation value $< x >$ evaluated on this state, now, has a term which is oscillating in time, proportional to

$$\int dx\ \psi_1(x)^* \cdot \psi_2(x) \cdot x$$

Some relevant properties of the time independent solutions ψ

1. Let $\psi(x)$ be a solution of the time independent Schrödinger equation for the energy E: we have
$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}\Big[V(x) - E\Big]\psi(x) \qquad (1)$$
2. This implies that $\psi(x)$ **admits the second derivative**, therefore $\frac{d\psi}{dx}$ and ψ must be differentiable and, therefore, also **continuous** functions.
3. If the potential energy $V(x)$ is differentiable ($\Rightarrow$ $F(x) \equiv -\frac{dV}{dx}$ if a regular function...), from eq.(1) we conclude that also $\frac{d^2\psi}{dx^2}$ is differentiable, and, therefore, continuous.

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1. Assume, now, that $V(x)$ changes abruptly between $x_0 - \epsilon$ and $x_0 + \epsilon$ by a relevant quantity ΔV_0. Consider the identity

$$\int_{x_0-\epsilon}^{x_0+\epsilon} dx \, \frac{d^2\psi}{dx^2} = \int_{x_0-\epsilon}^{x_0+\epsilon} dx \, \frac{2m}{\hbar^2} \Big[V(x) - E \Big] \psi(x)$$

2. The integral on the left side is the difference

$$\int_{x_0-\epsilon}^{x_0+\epsilon} dx \, \frac{d^2\psi}{dx^2} = \left. \frac{d\psi}{dx} \right|_{(x_0+\epsilon)} - \left. \frac{d\psi}{dx} \right|_{(x_0-\epsilon)}$$

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2. However, if the discontinuity in $V(x)$ is infinite, we will see later that only ψ remains continuous ...
3. In conclusion:

$$V(x)\ regular: \ \Rightarrow \ \psi, \frac{d\psi}{dx}, \frac{d^2\psi}{dx^2}\ continuous$$

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