QUANTUM MECHANICS Lecture 6

Enrico Iacopini

QUANTUM MECHANICS Lecture 6 Still about the Schrödinger equation

Enrico Iacopini

September 18, 2019

D. J. Griffiths: paragraph 2.2

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General solution of the Schrödinger equation

- In the previous lecture we have said that the procedure to find the **general solution** of **the time dependent** Schrödinger equation, for an initial condition $\Psi(x, 0)$, is as follows.
- 2 We solve the time-independent Schrödinger equation, which, in general, has infinite solutions $\psi_1(x), ..., \psi_n(x), ...$ corresponding to different energies $E_1, ..., E_n, ...$
- (a) We write $\Psi(x, 0)$ as a linear combination of the above stationary solutions, i.e.

$$\Psi(x,0) = \sum_n c_n \, \psi_n(x)$$

where the c_n are suitable complex

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General solution of Schrödinger equation

We define the function

$$\Psi(x,t) = \sum_n c_n \, e^{-i E_n t/\hbar} \, \psi_n(x)$$

Since it is a linear combination of solutions of the *time – dependent* Schrödinger equation, it is certainly one of its possible solutions.

3 At t = 0, the w.f. $\Psi(x, t)$ satisfies the initial condition that we have imposed, therefore it is the solution that we were looking for, because the solution with a given initial condition is unique.

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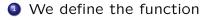
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Let us come again, now, to the argument of the time dependence af the quantum observables

- We have seen that each stationary state describes a physical state which appears "frozen" in time: no time dependence of any physical quantity !
- But things do change in time ... how it happens ?

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Suppose that

$$\Psi(x,0) = c_1 \,\psi_1(x) + c_2 \,\psi_2(x)$$

where, for sake of simplicity, c_1 and c_2 are real numbers such that $c_1^2 + c_2^2 = 1$ and ψ_1 , ψ_2 are real *normalized* (orthogonal) stationary solutions, corresponding to the energies E_1 and E_2 , with $E_1 \neq E_2$.

2 Then the time-dependent solution reads

$$\begin{split} \Psi(x,t) &= c_1 \, \psi_1(x) \, e^{-iE_1 t/\hbar} + c_2 \, \psi_2(x) \, e^{-iE_2 t/\hbar} \equiv \\ &\equiv c_1 \Psi_1(x,t) + c_2 \Psi_2(x,t) \end{split}$$

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Let us see, for instance, how it behaves in time the expectation value of the position x.

If we define $\Delta E \equiv E_2 - E_1$, then we have

$$egin{array}{rcl} < x > &=& \int dx \, (c_1 \Psi_1 + c_2 \Psi_2)^* \, x \, (c_1 \Psi_1 + c_2 \Psi_2) = \ &=& c_1^2 < x >_1 + c_2^2 < x >_2 + \ &+& 2c_1 \, c_2 \, \cos \left(rac{\Delta Et}{\hbar}
ight) \int dx \, \, \psi_1(x)^* \cdot \psi_2(x) \cdot x \end{array}$$

which shows that the expectation value $\langle x \rangle$ evaluated on this state, now, has a term which is oscillating in time, proportional to

$$\int dx \,\,\psi_1(x)^*\cdot\psi_2(x)\cdot x$$

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I Let $\psi(x)$ be a solution of the time independent Schrödinger equation for the energy *E*: we have

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2} \Big[V(x) - E \Big] \psi(x) \tag{1}$$

This implies that $\psi(x)$ admits the second derivative, therefore $\frac{d\psi}{dx}$ and ψ must be differentiable and, therefore, also continuous functions.

If the potential energy V(x) is differentiable $(\Rightarrow F(x) \equiv -\frac{dV}{dx}$ if a regular function...), from eq.(1) we conclude that also $\frac{d^2\psi}{dx^2}$ is differentiable, and, therefore, continuous.

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Assume, now, that V(x) changes abruptly between $x_0 - \epsilon$ and $x_0 + \epsilon$ by a relevant quantity ΔV_0 . Consider the identity

$$\int_{x_0-\epsilon}^{x_0+\epsilon} dx \, rac{d^2 \psi}{dx^2} = \int_{x_0-\epsilon}^{x_0+\epsilon} dx \, rac{2m}{\hbar^2} \Big[V(x) - E \Big] \psi(x)$$

The integral on the left side is the difference

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Solution The integrand at the right side is limited and, therefore, the integral goes to zero when $\epsilon \rightarrow 0$.

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- This means that, also if V presents at $x = x_0$ a step of **finite** amplitude ΔV_0 , $\frac{d\psi}{dx}$ remains continuous in $x = x_0$ (and the same for ψ !).
- 2 However, if the discontinuity in V(x) is infinite, we will see later that only ψ remains continuous . . .
- In conclusion:

 $V(x) regular : \Rightarrow \psi, \frac{d\psi}{dx}, \frac{d^2\psi}{dx^2} continuous$ $\Delta V_0 finite : \Rightarrow \psi, \frac{d\psi}{dx} continuous$ $\Delta V_0 infinite : \Rightarrow \psi continuous$

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