Exercise N.6

Exercise

In a bidimensional Hilbert space, $\mathbf{e}_1$ and $\mathbf{e}_2$ form an orthonormal basis.

- Consider the system of the two vectors $\mathbf{a}_1 \equiv \mathbf{e}_1 + i\,\mathbf{e}_2$ and $\mathbf{a}_2 \equiv i\,\mathbf{e}_1 - \mathbf{e}_2$.
  Do $\mathbf{a}_1$ and $\mathbf{a}_2$ form a basis ? Explain.
- Show that

$$\mathbf{f}_1 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2); \quad \mathbf{f}_2 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 - \mathbf{e}_2)$$

  form an orthonormal basis.
- Write the 2×2 matrix $\mathbf{A}$ that allow to express the vectors $\mathbf{f}_i$ in terms of the vectors $\mathbf{e}_j$, i.e. $\mathbf{f}_i = \mathbf{A}_{ji}\mathbf{e}_j$; $i, j = 1, 2$.

Solution

1. The two vectors $\mathbf{a}_1 \equiv \mathbf{e}_1 + i\,\mathbf{e}_2$ and $\mathbf{a}_2 \equiv i\,\mathbf{e}_1 - \mathbf{e}_2$ are proportional, in fact $\mathbf{e}_2 = i\,\mathbf{e}_1$ therefore they do not form a basis.
2. Concerning the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$, we have

$$\begin{aligned} < \mathbf{f}_1|\mathbf{f}_1 > &= \frac{1}{2}(< \mathbf{e}_1|\mathbf{e}_1 > + < \mathbf{e}_1|\mathbf{e}_2 > + \\ &+ < \mathbf{e}_2|\mathbf{e}_1 > + < \mathbf{e}_2|\mathbf{e}_2 >) = \\ &= \frac{1}{2}(1+0+0+1) = 1 \\ < \mathbf{f}_2|\mathbf{f}_2 > &= \frac{1}{2}(< \mathbf{e}_1|\mathbf{e}_1 > - < \mathbf{e}_1|\mathbf{e}_2 > - \\ &- < \mathbf{e}_2|\mathbf{e}_1 > + < \mathbf{e}_2|\mathbf{e}_2 >) = \\ &= \frac{1}{2}(1-0-0+1) = 1 \end{aligned}$$

Solution

1. The two vectors $\mathbf{a}_1 \equiv \mathbf{e}_1 + i\,\mathbf{e}_2$ and $\mathbf{a}_2 \equiv i\,\mathbf{e}_1 - \mathbf{e}_2$ are proportional, in fact $\mathbf{e}_2 = i\,\mathbf{e}_1$ therefore they do not form a basis.
2. Concerning the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$, we have

$$
\begin{aligned}
< \mathbf{f}_1|\mathbf{f}_1 > &= \frac{1}{2}(< \mathbf{e}_1|\mathbf{e}_1 > + < \mathbf{e}_1|\mathbf{e}_2 > + \\
&+ < \mathbf{e}_2|\mathbf{e}_1 > + < \mathbf{e}_2|\mathbf{e}_2 >) = \\
&= \frac{1}{2}\,(1 + 0 + 0 + 1) = 1 \\
< \mathbf{f}_2|\mathbf{f}_2 > &= \frac{1}{2}(< \mathbf{e}_1|\mathbf{e}_1 > - < \mathbf{e}_1|\mathbf{e}_2 > - \\
&- < \mathbf{e}_2|\mathbf{e}_1 > + < \mathbf{e}_2|\mathbf{e}_2 >) = \\
&= \frac{1}{2}\,(1 - 0 - 0 + 1) = 1
\end{aligned}
$$

Solution

$$
\begin{aligned}
< \mathbf{f}_1|\mathbf{f}_2 > &= \frac{1}{2}(< \mathbf{e}_1|\mathbf{e}_1 > - < \mathbf{e}_1|\mathbf{e}_2 > + \\
&+ < \mathbf{e}_2|\mathbf{e}_1 > - < \mathbf{e}_2|\mathbf{e}_2 >) = \\
&= \frac{1}{2}\,(1 - 0 + 0 - 1) = 0
\end{aligned}
$$

$$
< \mathbf{f}_2|\mathbf{f}_1 > \;=\; < \mathbf{f}_1|\mathbf{f}_2 >^* = 0
$$

so they do form an orthonormal basis.

Solution

Since, by definition, we have

$$\mathbf{f}_1 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2); \quad \mathbf{f}_2 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 - \mathbf{e}_2)$$

the matrix $\mathbf{A}$ for which $\mathbf{f}_i = \mathbf{A}_{ji}\mathbf{e}_j$ reads

$$\mathbf{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$