Exercise N.3

Exercise

Evaluate the expectation values $< x >$ and $< p >$, together with the standard deviations σ_x and σ_p on the physical state represented by the w.f. $\Psi_n(x)$

$$\psi_n(x) = \sqrt{\frac{2}{a}}\, sin\left(\frac{n\pi x}{a}\right) \quad for \quad 0 \leq x \leq a$$

and verify the uncertainty relation.

Solution

Since all the ψ_n have a **definite parity** with respect to $a/2$, $|\psi|^2$ is **even** with respect to $a/2$ and, therefore $< x >= \frac{a}{2}$:

$$< x >= \int_0^a dx \, |\psi_n(x)|^2 \cdot x = \frac{a}{2}$$

Solution

1. Since ψ_n describes a stationary state, $< x >$ is **time-independent** ($= a/2$, as we have already seen ...), and therefore

$$< p > = m \frac{d}{dt} < x > = 0$$

2. Directly, we have

$$\begin{aligned} < p > &= -i\hbar \int dx \, \psi_n^* \left(\frac{d\psi_n}{dx} \right) = \\ &= -i\hbar \, \frac{2}{a} \int_0^a dx \, sin \left(\frac{n\pi x}{a} \right) \frac{n\pi}{a} \, cos \left(\frac{n\pi x}{a} \right) \\ &= -i \frac{\hbar n\pi}{a^2} \int_0^a dx \;\; sin \left(\frac{2n\pi x}{a} \right) \end{aligned}$$

and the integral, being made on an integer number n of periods, is always equal to zero.

Solution

1. Since ψ_n describes a stationary state, $< x >$ is **time-independent** ($= a/2$, as we have already seen ...), and therefore

$$< p >= m\frac{d}{dt} < x >= 0$$

2. Directly, we have

$$\begin{aligned} < p > &= -i\hbar \int dx\, \psi_n^* \left(\frac{d\psi_n}{dx}\right) = \\ &= -i\hbar\, \frac{2}{a} \int_0^a dx\, sin\left(\frac{n\pi x}{a}\right) \frac{n\pi}{a}\, cos\left(\frac{n\pi x}{a}\right) \\ &= -i\frac{\hbar n\pi}{a^2} \int_0^a dx\;\, sin\left(\frac{2n\pi x}{a}\right) \end{aligned}$$

and the integral, being made on an integer number n of periods, is always equal to zero.

Solution

Let us, now, evaluate $< x^2 >$. We have

$$< x^2 > = \int dx \, |\psi_n|^2 \cdot x^2 = \frac{2}{a} \int_0^a dx \, x^2 \, sin^2\left(\frac{n\pi x}{a}\right) =$$

$$= \frac{2}{a} \int_0^a dx \, x^2 \frac{1}{2} \left[1 - cos\left(\frac{2n\pi x}{a}\right)\right] =$$

$$= \frac{1}{a} \int_0^a dx \, x^2 - \frac{1}{a} \int_0^a dx \, x^2 \, cos\left(\frac{2n\pi x}{a}\right)$$

The first term is obviously equal to $\frac{a^2}{3}$, whereas the second one, after a double integration by parts (see later), is equal to $\frac{a^2}{2(n\pi)^2}$. Then

$$\sigma_x^2 = < x^2 > - < x >^2 = \frac{a^2}{3} - \frac{a^2}{2(n\pi)^2} - \frac{a^2}{4} =$$

$$= a^2 \left(\frac{1}{12} - \frac{1}{2(n\pi)^2}\right) \Rightarrow \sigma_x = a\sqrt{\frac{1}{12} - \frac{1}{2(n\pi)^2}}$$

Solution

Concerning $< p^2 >$, we notice that $p^2 = 2m\,H$ where H is the hamiltonian. Therefore, on the w.f. ψ_n we will have

$$< p^2 >= 2m\,E_n = 2m\,\frac{1}{2m}\left(\frac{n\pi\hbar}{a}\right)^2 = \left(\frac{n\pi\hbar}{a}\right)^2$$

But we already said that $< p >= 0$, therefore

$$\sigma_p^2 =< p^2 > - < p >^2 =< p^2 > \;\Rightarrow\; \sigma_p = \hbar\frac{n\pi}{a}$$

$$\Rightarrow \sigma_x\,\sigma_p = a\sqrt{\frac{1}{12} - \frac{1}{2(n\pi)^2}}\;\hbar\frac{n\pi}{a} = \hbar\sqrt{\frac{(n\pi)^2}{12} - \frac{1}{2}} \geq$$

$$\geq 0.5679\,\hbar\;\;(at\;\;n = 1) > \frac{\hbar}{2}$$

Solution

Let us explicitly show, now, that

$$\frac{1}{a}\int_0^a dx\, x^2 \cos\left(\frac{2n\pi x}{a}\right) = \frac{a^2}{2(n\pi)^2}$$

In fact, the integrand can be rewritten as

$$\begin{aligned} x^2 \cos\left(\frac{2n\pi x}{a}\right) &= \frac{d}{dx}\left[x^2 \sin\left(\frac{2n\pi x}{a}\right)\right]\frac{a}{2n\pi} - \\ &- \frac{a}{2n\pi}\, 2x \sin\left(\frac{2n\pi x}{a}\right) \end{aligned}$$

The contribute to the integral coming from the first term is simply

$$x^2 \sin\left(\frac{2n\pi x}{a}\right)\Big|_0^a = 0$$

Solution

therefore we remain with the integral

$$\frac{1}{a}\left(-\frac{a}{n\pi}\right)\int_0^a dx\, x\, sin\left(\frac{2n\pi x}{a}\right)$$

But, again, the integrand can be rewritten as

$$x\, sin\left(\frac{2n\pi x}{a}\right) = -\frac{d}{dx}\left[x\cos\left(\frac{2n\pi x}{a}\right)\right]\frac{a}{2n\pi} + + \frac{a}{2n\pi}\cos\left(\frac{2n\pi x}{a}\right)$$

and the contribution from the first term is what we said, i.e.

$$\frac{1}{a}\left(-\frac{a}{n\pi}\right)(-)\frac{a}{2n\pi}\left[x\cos\left(\frac{2n\pi x}{a}\right)\right]\Big|_0^a = \frac{a^2}{2(n\pi)^2}$$

whereas the second term does not contribute (integral of the cosine between 0 and $2n\pi$).