QUANTUM MECHANICS Appendix 1

<T> and <V> for the harmonic oscillator

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Appendix1: <T> and <V> for the harmonic oscillator

1. We have shown already that, on any **stationary state** ψ_n of a harmonic oscillator, we have

$$< T > = < V > = \frac{1}{2} < H > \equiv \frac{1}{2} E_n = \frac{\hbar\omega}{2}\left(n + \frac{1}{2}\right)$$

2. Let us show, now, that, on a **generic state**, one has

$$\begin{aligned} < T > &= \frac{1}{2} < H > + A\,cos(2\omega t + \phi) \\ < V > &= \frac{1}{2} < H > - A\,cos(2\omega t + \phi) \end{aligned}$$

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To prove the statement, let us start from the following definitions/results

$$
\begin{aligned}
a_\pm &= \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} \mp i\hat{p}) \\
\hat{p} &= i\sqrt{\frac{m\hbar\omega}{2}}(a_+ - a_-) \\
\hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \\
a_+a_- &= \frac{1}{\hbar\omega}(\hat{H} - \frac{1}{2}); \quad a_-a_+ = \frac{1}{\hbar\omega}(\hat{H} + \frac{1}{2}) \\
a_+\psi_n &= \sqrt{n+1}\psi_{n+1}; \quad a_-\psi_n = \sqrt{n}\psi n - 1
\end{aligned}
$$

Appendix1: <T> and <V> for the harmonic oscillator

We have already shown that

$$T = \frac{1}{2m}\hat{p}^2 =$$
$$= \frac{1}{2m}\frac{-m\hbar\omega}{2}(a_+a_+ - a_+a_- - a_-a_+ + a_-a_-) =$$
$$= \frac{\hbar\omega}{4}(a_+a_- + a_-a_+ - a_+a_+ - a_-a_-)$$

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but

$$a_+a_- + a_-a_+ = \frac{1}{\hbar\omega}\left[\left(\hat{H} - \frac{1}{2}\right) + \left(\hat{H} + \frac{1}{2}\right)\right] =$$
$$= \frac{2}{\hbar\omega}\,\hat{H}$$

and therefore

$$T = \frac{\hbar\omega}{4}\frac{2}{\hbar\omega}\,\hat{H} - \frac{\hbar\omega}{4}a_+a_+ - \frac{\hbar\omega}{4}a_-a_- \equiv$$
$$\equiv T1 + T2 + T3$$

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Let us suppose, now, that the state of the harmonic oscillator is described by the w.f.

$$\begin{aligned}\Psi(x,t) &= \sum_n c_n \psi_n(x) e^{-iE_n t/\hbar} = \\ &= e^{-i\omega t/2} \sum_n c_n \psi_n e^{-in\omega t}\end{aligned}$$

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Let us evaluate $< T >$ on this state, at time t.

The first term gives

$$\begin{aligned} < T1 > &= \int dx \sum_n c_n^* \psi_n^* e^{in\omega t} \frac{1}{2} H \sum_s c_s \psi_s e^{-is\omega t} = \\ &= \frac{1}{2} \sum_{n,s} e^{i\omega t(n-s)} c_n^* c_s E_s \int dx\, \psi_n^* \psi_s = \\ &= \frac{1}{2} \sum_n |c_n|^2 E_n \equiv \frac{1}{2} < \hat{H} > \end{aligned}$$

Appendix1: <T> and <V> for the harmonic oscillator

1. Let us consider, now, the second term $< T2 > \propto a_+ a_+$.

2. We have

$$< T2 > = -\frac{\hbar\omega}{4} \int dx \sum_n c_n^* \psi_n^* e^{in\omega t} a_+ a_+ \sum_s c_s \psi_s e^{-is\omega t} =$$

$$= -\frac{\hbar\omega}{4} \sum_{n,s} e^{i\omega t(n-s)} c_n^* c_s \int dx \, \psi_n^* (a_+ a_+ \psi_s)$$

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But

$$
\begin{aligned}
a_+a_+\psi_s &= a_+\sqrt{s+1}\psi_{s+1} = \\
&= \sqrt{(s+1)(s+2)}\psi_{s+2}
\end{aligned}
$$

and the orthonormality of the ψ_n guarantees that the integral is null if $s+2 \neq n$ or, in other words if $s \neq n-2$.

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Therefore

$$< T2 > \; = \; -\frac{\hbar\omega}{4}\sum_n c_n^* \, c_{n-2} e^{2i\omega t}\sqrt{n(n-1)} =$$

$$= \; -\frac{\hbar\omega}{4} e^{2i\omega t}\sum_n c_n^* \, c_{n-2}\sqrt{n(n-1)}$$

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Concerning the last term $< T3 > \propto a_- a_-$, we have

$$< T3 > = -\frac{\hbar\omega}{4} \int dx \sum_n c_n^* \psi_n^* e^{in\omega t} a_- a_- \sum_s c_s \psi_s e^{-is\omega t} =$$

$$= -\frac{\hbar\omega}{4} \sum_{n,s} e^{i\omega t(n-s)} c_n^* c_s \int dx \, \psi_n^* (a_- a_- \psi_s)$$

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But

$$\begin{aligned} a_- a_- \psi_s &= a_- \sqrt{s}\psi_{s-1} = \\ &= \sqrt{s(s-1)}\psi_{s-2} \end{aligned}$$

and the orthonormality of the ψ_n guarantees that the integral is null if $s-2 \neq n$ or, in other words if $s \neq n+2$; therefore

$$\begin{aligned} < T3 > &= -\frac{\hbar\omega}{4}\sum_s c^*_{s-2}\, c_s e^{-2i\omega t}\sqrt{s(s-1)} = \\ &= -\frac{\hbar\omega}{4} e^{-2i\omega t}\sum_s c^*_{s-2}\, c_s \sqrt{s(s-1)} \end{aligned}$$

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It is now evident that $< T3 > = < T2 >^*$;
therefore if we define

$$-\frac{\hbar\omega}{4}\sum_n c_n^* \, c_{n-2}\sqrt{n(n-1)} \equiv \frac{A}{2}\, e^{i\phi}$$

with $A \geq 0$ and ϕ reals, then

$$< T2 > \; = \; \frac{A}{2} e^{2i\omega t} e^{i\phi}$$
$$< T3 > \; = \; \frac{A}{2} e^{-2i\omega t} e^{-i\phi}$$

and, therefore

$$< T2 > + < T3 > = A\, cos(\omega t + \phi)$$

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In conclusion

$$
\begin{aligned}
< T > &= < T1 > + < T2 > + < T3 >= \\
&= \frac{1}{2} < \hat{H} > + A\, cos(2\omega t + \phi)
\end{aligned}
$$

$$
\begin{aligned}
< V > &= < \hat{H} > - < T >= \\
&= \frac{1}{2} < \hat{H} > - A\, cos(2\omega t + \phi)
\end{aligned}
$$

As in Classical Mechanics, the time dependence is at twice the harmonic oscillator frequency and the time average of $< T >$ and $< V >$ are both equal to $\frac{1}{2} < \hat{H} > \equiv \frac{1}{2} E$.

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